

# MOP 2019 Red Group Tests

Compiled by Eric Shen

## Contents

<b>1 Problems</b>	<b>1</b>
1.1 MOP Test 1	1
1.2 MOP Test 2	2
1.3 MOP Test 3	2
1.4 MOP Test 4	2
<b>2 Solutions</b>	<b>3</b>
2.1 Solutions to MOP Test 1	3
2.2 Solutions to MOP Test 2	6
2.3 Solutions to MOP Test 3	8
2.4 Solutions to MOP Test 4	10

## §1 Problems

### §1.1 MOP Test 1

**Problem R1.1.** Let  $n$  be a positive integer. Determine the number of sequences of real numbers  $a_1, \dots, a_n$  such that

$$a_{k+1} = a_k^2 + a_k - 1$$

for all  $k$  (with indices modulo  $n$ ).

**Problem R1.2.** Let  $n > 1$  be a positive integer. Each cell of an  $n \times n$  table contains an integer which is  $1 \pmod n$ . It is given that the sum of the numbers in any row, as well as the sum of numbers in any column, is congruent to  $n \pmod{n^2}$ . Denote by  $R_i$  the product of the numbers in the  $i^{\text{th}}$  row and by  $C_j$  the product of the numbers in the  $j^{\text{th}}$  column. Prove that

$$R_1 + \dots + R_n \equiv C_1 + \dots + C_n \pmod{n^4}.$$

**Problem R1.3.** Let  $n \geq 2$  be an integer. There are  $n^2$  marbles, each of which is one of  $n$  colors, but not necessarily  $n$  of each color. Prove that they may be placed in  $n$  boxes with  $n$  marbles in each box, such that each box contains marbles of at most two colors.

**Problem R1.4** (ISL 2018 G5). Let  $ABC$  be a triangle with circumcircle  $\omega$  and incenter  $I$ . A line  $\ell$  meets the lines  $AI$ ,  $BI$ , and  $CI$  at points  $D$ ,  $E$ , and  $F$ , respectively, all distinct from  $A$ ,  $B$ ,  $C$  and  $I$ . Prove that the circumcircle of the triangle determined by the perpendicular bisectors of  $\overline{AD}$ ,  $\overline{BE}$ , and  $\overline{CF}$  is tangent to  $\omega$ .

## §1.2 MOP Test 2

**Problem R2.1.** Diagonals  $\overline{AC}$  and  $\overline{BD}$  of convex quadrilateral  $ABCD$  meet at  $P$ . Prove that the incenters of triangles  $\triangle PAB$ ,  $\triangle PBC$ ,  $\triangle PCD$ , and  $\triangle PDA$  are concyclic if and only if their  $P$ -excenters are concyclic.

**Problem R2.2.** Let  $n \geq 3$  be an integer. Prove that there exists a set  $S$  of  $2n$  distinct positive integers such that for any  $m \in \{2, \dots, n\}$ , the set  $S$  can be partitioned into two sets with cardinalities  $m$  and  $2n - m$  with equal sums.

**Problem R2.3.** Let  $a, b, c, d, x$  be positive integers obeying

$$x^2 - (a^2 + b^2 + c^2 + d^2 + 1)x + (a + c)(b + d) = 0.$$

Prove that  $x$  is a perfect square.

**Problem R2.4.** Let  $k$  be a positive integer and consider a  $3 \times 3 \times 3$  cube. We paint some of the 27 cells ultraviolet, such that in any  $2 \times 2 \times 1$  block,  $2 \times 1 \times 2$  block, or  $1 \times 2 \times 2$  block, at most  $k$  of the cells of the ultraviolet. Find the maximum possible number of ultraviolet cells in terms of  $k$ .

## §1.3 MOP Test 3

**Problem R3.1.** The numbers  $1, 2, 3, \dots, 1024$  are written on a blackboard. The following procedure is performed ten times: partition the numbers on the board into disjoint pairs, and replace each pair with its nonnegative difference. Determine all possible values of the final number.

**Problem R3.2.** Let  $\mathbb{Q}_{>0}$  denote the positive rational numbers. Find all functions  $f : \mathbb{Q}_{>0} \rightarrow \mathbb{Q}_{>0}$  obeying

$$f(x^2 f(y)^2) = f(x)^2 f(y)$$

for all positive rational numbers  $x$  and  $y$ .

**Problem R3.3.** Determine whether there exists a sequence  $a_1, a_2, a_3, \dots$  of nonnegative reals such that

$$a_n + a_{2n} + a_{3n} + \dots = \frac{1}{n}$$

for every integer  $n$ .

**Problem R3.4.** Let  $ABC$  be an acute triangle with circumcircle  $\Gamma$  and altitudes  $\overline{AD}$ ,  $\overline{BE}$ , and  $\overline{CF}$  meeting at  $H$ . Let  $\omega$  be the circumcircle of  $\triangle DEF$ . Point  $S \neq A$  lies on  $\Gamma$  such that  $DS = DA$ . Line  $\overline{AD}$  meets  $\overline{EF}$  at  $Q$ , and meets  $\omega$  at  $L \neq D$ . Point  $M$  is chosen such that  $\overline{DM}$  is a diameter of  $\omega$ . Point  $P$  lies on  $\overline{EF}$  with  $\overline{DP} \perp \overline{EF}$ . Prove that lines  $SH$ ,  $MQ$ , and  $PL$  are concurrent.

## §1.4 MOP Test 4

**Problem R4.1.** Let  $a_j, b_j, c_j$  be integers for  $1 \leq j \leq N$ . Assume for each  $j$ , at least one of  $a_j, b_j, c_j$  is odd. Show that there exist integers  $r, s, t$  such that  $ra_j + sb_j + tc_j$  is odd for at least  $\frac{4N}{7}$  values of  $j$ ,  $1 \leq j \leq N$ .

**Problem R4.2.** Let  $n \geq 3$  be a fixed integer. An  $n \times n$  table of nonzero real numbers is given with the property that for any cell, the sum of the  $2n - 2$  numbers in the other cells of the same row or column is exactly  $k$  times the number in that cell. Determine all possible values of  $k$ , in terms of  $n$ .

**Problem R4.3.** Let  $ABC$  be a triangle with  $AB = AC$ , and let  $M$  be the midpoint of  $\overline{BC}$ . Let  $P$  be a point such that  $PB < PC$  and  $\overline{PA} \parallel \overline{BC}$ . Let  $X$  and  $Y$  be points on lines  $PB$  and  $PC$  such that  $B$  lies inside segment  $PX$ ,  $C$  lies inside segment  $PY$ , and  $\angle PXM = \angle PYM$ . Prove that quadrilateral  $APXY$  is cyclic.

**Problem R4.4.** Determine all functions  $f : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  satisfying

$$\left(x + \frac{1}{x}\right) f(y) = f(xy) + f\left(\frac{y}{x}\right)$$

for all  $x, y > 0$ .

## §2 Solutions

### §2.1 Solutions to MOP Test 1

#### Problem R1.1 (Mexico 2011/3)

Let  $n$  be a positive integer. Determine the number of sequences of real numbers  $a_1, \dots, a_n$  such that

$$a_{k+1} = a_k^2 + a_k - 1$$

for all  $k$  (with indices modulo  $n$ ).

The answer is 2. I claim that only  $a_1 = 1$  and  $a_1 = -1$  work. Sum both sides to obtain

$$\sum_{k=1}^n a_{k+1} = \sum_{k=1}^n (a_k^2 + a_k - 1) \implies \sum_{k=1}^n a_k^2 = n.$$

Rewrite each term as  $a_{k+1} + 1 = a_k(a_k + 1)$ , and multiply to obtain

$$\prod_{k=1}^n (a_{k+1} + 1) = \prod_{k=1}^n a_k(a_k + 1) \implies \prod_{k=1}^n a_k = 1 \text{ or } a_k = -1.$$

If  $a_k = -1$  for some  $k$ , then the sequence is constant at  $-1$ . Otherwise, by AM-GM,

$$1 = \frac{1}{n} \sum_{k=1}^n a_k^2 \geq \prod_{k=1}^n a_k = 1,$$

so we are in the equality case, and  $a_k = \pm 1$ , which implies that the sequence is all 1, as desired.

#### Problem R1.2 (ISL 2018 N2)

Let  $n > 1$  be a positive integer. Each cell of an  $n \times n$  table contains an integer which is  $1 \pmod{n}$ . It is given that the sum of the numbers in any row, as well as the sum of numbers in any column, is congruent to  $n \pmod{n^2}$ . Denote by  $R_i$  the product of the numbers in the  $i^{\text{th}}$  row and by  $C_j$  the product of the numbers in the  $j^{\text{th}}$  column. Prove that

$$R_1 + \dots + R_n \equiv C_1 + \dots + C_n \pmod{n^4}.$$

Let  $na_{i,j} + 1$  denote the number in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column, so that  $\sum_{k=1}^n a_{i,k} = \sum_{k=1}^n a_{k,j} = 1$  for all  $k$ . Denote by  $P$  the product of all the numbers in the table. Check that

$$R_i \equiv \prod_{k=1}^n (na_{i,k} + 1) \equiv \left( n \sum_{k=1}^n a_{i,k} \right) + 1 \equiv 1 \pmod{n^2},$$

and similarly  $C_j \equiv 1 \pmod{n^2}$ . Notice that

$$P \equiv \prod_{i=1}^n R_i \equiv \prod_{i=1}^n ((R_i - 1) + 1) \equiv 1 + \sum_{i=1}^n (R_i - 1) \equiv \left( \sum_{i=1}^n R_i \right) - (n - 1) \pmod{n^4},$$

whence

$$\sum_{i=1}^n R_i \equiv P + (n - 1) \equiv \sum_{j=1}^n C_j \pmod{n^4},$$

as desired.

**Problem R1.3**

Let  $n \geq 2$  be an integer. There are  $n^2$  marbles, each of which is one of  $n$  colors, but not necessarily  $n$  of each color. Prove that they may be placed in  $n$  boxes with  $n$  marbles in each box, such that each box contains marbles of at most two colors.

We claim this construction works. At each step:

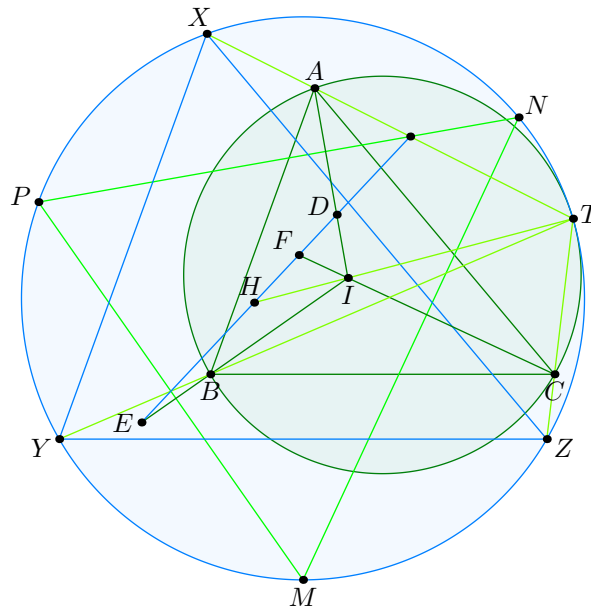
- Take as many of the color with least number of remaining marbles, and place them in a box.
- Fill up the rest of the box with marbles of the color with the most number of remaining marbles.

Suppose at a certain step  $c$  boxes remain,  $cn$  marbles remain, and  $c$  colors remain. By the Pigeonhole Principle, there is a color with at most  $n$  marbles, so we can select that color and use it up completely. There is also color with at least  $n$  marbles, so we may fill up any remaining space with it. Hence this process works inductively.

**Problem R1.4 (ISL 2018 G5)**

Let  $ABC$  be a triangle with circumcircle  $\omega$  and incenter  $I$ . A line  $\ell$  meets the lines  $AI$ ,  $BI$ , and  $CI$  at points  $D$ ,  $E$ , and  $F$ , respectively, all distinct from  $A$ ,  $B$ ,  $C$  and  $I$ . Prove that the circumcircle of the triangle determined by the perpendicular bisectors of  $\overline{AD}$ ,  $\overline{BE}$ , and  $\overline{CF}$  is tangent to  $\omega$ .

**First solution, by angle chasing.** Let  $\ell = \overline{DEF}$  and  $\ell_A$ ,  $\ell_B$ , and  $\ell_C$  denote its reflections across the perpendicular bisectors of  $\overline{AD}$ ,  $\overline{BE}$ , and  $\overline{CF}$ , respectively.



**Claim.** Lines  $\ell_A$ ,  $\ell_B$ , and  $\ell_C$  concur at a point  $T$  on  $\omega$ .

*Proof.* Check that

$$\angle(\ell_B, \ell_C) = \angle(\ell_B, \ell) + \angle(\ell, \ell_C) = 2\angle(\ell_B, \overline{BI}) + 2\angle(\overline{CI}, \ell_C) = 2\angle(\ell_B, \ell_C) + 2\angle CIB,$$

whence  $\angle(\ell_B, \ell_C) = 2\angle BIC = \angle BAC$ , as desired.  $\square$

Now let  $H = \overline{TI} \cap \ell$ . A homothety at  $T$  maps  $I$  to  $H$  and  $\triangle ABC$  to some triangle  $\triangle XYZ$  whose incenter is  $H$  such that  $(ABC)$  and  $(XYZ)$  are tangent at  $T$ .

However,  $\overline{XH} \parallel \overline{AD}$ , so the perpendicular bisector of  $\overline{XH}$  is the perpendicular bisector of  $\overline{AD}$ , whence the vertices of the triangle formed by the perpendicular bisectors of  $\overline{AD}$ ,  $\overline{BE}$ , and  $\overline{CF}$  are the arc midpoints of  $\triangle XYZ$ , which completes the proof.

**Second solution, by Simson Lines** (Pitchayut Saengrungkongka). Let the midpoints of arcs  $BC$ ,  $CA$ ,  $AB$  not containing  $A$ ,  $B$ ,  $C$  be  $M_A$ ,  $M_B$ ,  $M_C$  respectively, and let lines through  $D$ ,  $E$ ,  $F$  perpendicular to  $\overline{AI}$ ,  $\overline{BI}$ ,  $\overline{CI}$ , respectively form a triangle  $A_1B_1C_1$ . Also let  $XYZ$  be the triangle described in the problem. Since  $\triangle M_A M_B M_C$  and  $\triangle XYZ$  are homothetic,  $\overline{XM_A}$ ,  $\overline{YM_B}$ ,  $\overline{ZM_C}$  concur at a point  $T$ .

**Claim.**  $IA_1B_1C_1$  and  $TM_A M_B M_C$  are homothetic.

*Proof.* Since  $\triangle A_1B_1C_1$  and  $\triangle M_A M_B M_C$  are homothetic it suffices to show that  $\overline{IA_1} \parallel \overline{M_A X}$ . Let  $I_A$  be the  $A$ -excenter. Then  $\overline{M_A X}$  is the  $I_A$ -midsegment of  $\triangle I_A I A_1$ , since the projections of  $I_A$ ,  $A_1$ ,  $X$  onto  $\overline{BI}$  are  $B$ , the midpoint of  $\overline{BE}$ , and  $E$ . Thus the claim is true.  $\square$

To finish, note that  $\overline{DEF}$  is the Simson Line from  $I$  to  $\triangle A_1B_1C_1$ , so  $IA_1B_1C_1$  and thus  $TM_A M_B M_C$  is cyclic. By homothety,  $(ABC)$  and  $(XYZ)$  are tangent at  $T$ , so we are done.

## §2.2 Solutions to MOP Test 2

### Problem R2.1 (India TST 2015/1)

Diagonals  $\overline{AC}$  and  $\overline{BD}$  of convex quadrilateral  $ABCD$  meet at  $P$ . Prove that the incenters of triangles  $\triangle PAB$ ,  $\triangle PBC$ ,  $\triangle PCD$ , and  $\triangle PDA$  are concyclic if and only if their  $P$ -excenters are concyclic.

The pith of this problem is this lemma:

#### Lemma

If  $I$  and  $I_A$  are the incenter and  $A$ -excenter of a triangle  $ABC$ , then  $AB \cdot AC = AI \cdot AI_A$ .

*Proof.* Since  $\angle BAI = \angle I_A AC$  and  $\angle ABI = \angle IBC = \angle I I_A C = \angle A I_A C$ ,  $\triangle ABI \sim \triangle A I_A C$  by AA, whence

$$\frac{AB}{AI} = \frac{AI_A}{AC} \implies AB \cdot AC = AI \cdot AI_A,$$

as requested. □

Applying the lemma,

$$(PI_1 \cdot PI_3) \cdot (PJ_1 \cdot PJ_3) = PA \cdot PB \cdot PC \cdot PD = (PI_2 \cdot PI_4) \cdot (PJ_2 \cdot PJ_4),$$

and the desired result is obvious.

### Problem R2.2 (ISL 2018 C1)

Let  $n \geq 3$  be an integer. Prove that there exists a set  $S$  of  $2n$  distinct positive integers such that for any  $m \in \{2, \dots, n\}$ , the set  $S$  can be partitioned into two sets with cardinalities  $m$  and  $2n - m$  with equal sums.

We claim that this generalizable set works:

$$S = \{1, 11, 111, 1111, 11111, \\ 12, 123, 1234, a, b\},$$

where  $a$  and  $b$  are chosen such that the sum of the first five elements equals the sum of the latter half.

Now, consider the subset  $\{1, 11, 111, 1111, 11111\}$ . To reduce the size, coalesce the two smallest terms. For instance, the next largest subset is  $\{12, 111, 1111, 11111\}$ , all the way to  $\{1234, 11111\}$ , so this works.

### Problem R2.3 (Romania 2007/9/1)

Let  $a, b, c, d, x$  be positive integers obeying

$$x^2 - (a^2 + b^2 + c^2 + d^2 + 1)x + (a + c)(b + d) = 0.$$

Prove that  $x$  is a perfect square.

Notice that by AM-GM,

$$a^2 + b^2 + c^2 + d^2 = \sum_{\text{cyc}} \frac{a^2 + b^2}{2} \geq ab + bc + cd + da = (a + c)(b + d).$$

Since the polynomial has integer roots, and the linear coefficient exceeds the constant term, one root must be 1, and we must be in the equality case. Hence,  $a = b = c = d$ , and the other root is clearly  $4a^2$ , so we are done.

**Problem R2.4**

Let  $k$  be a positive integer and consider a  $3 \times 3 \times 3$  cube. We paint some of the 27 cells ultraviolet, such that in any  $2 \times 2 \times 1$  block,  $2 \times 1 \times 2$  block, or  $1 \times 2 \times 2$  block, at most  $k$  of the cells of the ultraviolet. Find the maximum possible number of ultraviolet cells in terms of  $k$ .

The answer is  $6k + 3$ . We split the cube into six  $2 \times 2 \times 1$  blocks and three extra cubes:

$$\begin{bmatrix} \bullet & A & A \\ F & F & B \\ F & F & B \end{bmatrix} \quad \begin{bmatrix} E & A & A \\ E & \bullet & B \\ D & D & B \end{bmatrix} \quad \begin{bmatrix} E & C & C \\ E & C & C \\ D & D & \bullet \end{bmatrix}$$

Each block can contain at most  $k$  ultraviolet cells, so an upper bound is  $6k + 3$ .

We now present equality cases. For  $k = 1$ ,

$$\begin{bmatrix} u & \cdot & u \\ \cdot & \cdot & \cdot \\ u & \cdot & u \end{bmatrix} \quad \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & u & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} u & \cdot & u \\ \cdot & \cdot & \cdot \\ u & \cdot & u \end{bmatrix}$$

For  $k = 2$ ,

$$\begin{bmatrix} u & \cdot & u \\ \cdot & u & \cdot \\ u & \cdot & u \end{bmatrix} \quad \begin{bmatrix} u & \cdot & u \\ \cdot & u & \cdot \\ u & \cdot & u \end{bmatrix} \quad \begin{bmatrix} u & \cdot & u \\ \cdot & u & \cdot \\ u & \cdot & u \end{bmatrix}$$

For  $k = 3$ ,

$$\begin{bmatrix} u & \cdot & u \\ u & u & u \\ u & \cdot & u \end{bmatrix} \quad \begin{bmatrix} u & u & u \\ \cdot & u & \cdot \\ u & u & u \end{bmatrix} \quad \begin{bmatrix} u & \cdot & u \\ u & u & u \\ u & \cdot & u \end{bmatrix}$$

We are done.

### §2.3 Solutions to MOP Test 3

#### Problem R3.1 (Tuymaada 2018 J6)

The numbers  $1, 2, 3, \dots, 1024$  are written on a blackboard. The following procedure is performed ten times: partition the numbers on the board into disjoint pairs, and replace each pair with its nonnegative difference. Determine all possible values of the final number.

The answer is all nonnegative evens less than 1024. Note that after one move, the largest possible number on the board is 1023, and thereafter, the largest number on the board is nondecreasing, so the last number does not exceed 1023. Furthermore, the sum of the numbers on the board is invariant, so the last number must be even.

Consider this construction: Partition the numbers into

$$(1, 2k+2), (2, 3), (4, 5), \dots, (2k, 2k+1), \\ (2k+3, 2k+4), \dots, (1023, 1024),$$

thence the board now reads  $2k+1, 1, 1, \dots, 1$ . It is now obvious that after one move, the board becomes  $2k, 0, 0, \dots, 0$ , and the numbers eventually coalesce into  $2k$ , as desired.

#### Problem R3.2 (ISL 2018 A1)

Let  $\mathbb{Q}_{>0}$  denote the positive rational numbers. Find all functions  $f : \mathbb{Q}_{>0} \rightarrow \mathbb{Q}_{>0}$  obeying

$$f(x^2 f(y)^2) = f(x)^2 f(y)$$

for all positive rational numbers  $x$  and  $y$ .

The answer is  $f \equiv 1$ . Let  $P(x, y)$  denote the assertion.  $P(f(1)^{-1}, 1)$  gives  $f(f(1)^{-1}) = 1$ , and  $P(x, f(1)^{-1})$  gives  $f(x^2) = f(x)^2$ .

Now, write

$$f(y) = \left[ \frac{f(xf(y))}{f(x)} \right]^2.$$

We can now show using induction on  $n$  that  $f(x)$  can be expressed as  $c^{2^n}$  for some  $c$ , so  $f \equiv 1$ , done.

#### Problem R3.3 (Iran 2001/3/2)

Determine whether there exists a sequence  $a_1, a_2, a_3, \dots$  of nonnegative reals such that

$$a_n + a_{2n} + a_{3n} + \dots = \frac{1}{n}$$

for every integer  $n$ .

The answer is no. Assume FSoC such a sequence exists, and let  $P_n$  denote the first  $n$  primes. By the Principle of Inclusion-Exclusion,

$$\frac{1}{n} - a_n = \lim_{k \rightarrow \infty} \sum_{T \subset P_k} \frac{(-1)^{|T|}}{n \prod_{p \in T} p} = \frac{1}{n} \lim_{k \rightarrow \infty} \sum_{T \subset P_k} \frac{(-1)^{|T|}}{\prod_{p \in T} p} = \frac{1 - a_1}{n}.$$

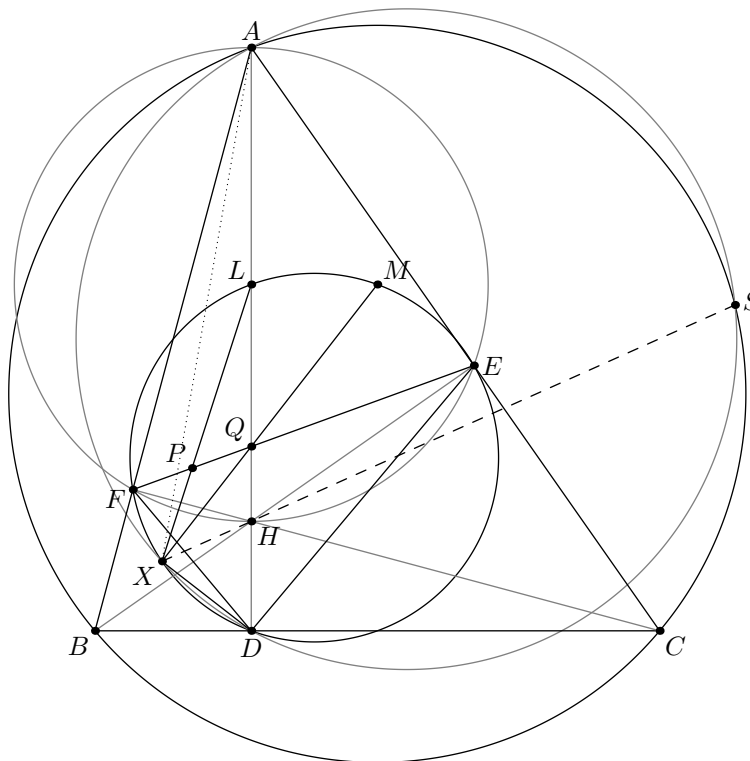
Solving,  $a_n = \frac{1}{n} a_1$  for all  $n$ , so

$$1 = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{a_1}{k} = a_1 \sum_{k=1}^{\infty} \frac{1}{k}.$$

Since  $\sum_{k=1}^{\infty} a_k$  diverges,  $a_1 = 0$ , which means that  $a_n = 0$  for all  $n$ . This yields the desired contradiction.

**Problem R3.4** (India TST 2019/8)

Let  $ABC$  be an acute triangle with circumcircle  $\Gamma$  and altitudes  $\overline{AD}$ ,  $\overline{BE}$ , and  $\overline{CF}$  meeting at  $H$ . Let  $\omega$  be the circumcircle of  $\triangle DEF$ . Point  $S \neq A$  lies on  $\Gamma$  such that  $DS = DA$ . Line  $\overline{AD}$  meets  $\overline{EF}$  at  $Q$ , and meets  $\omega$  at  $L \neq D$ . Point  $M$  is chosen such that  $\overline{DM}$  is a diameter of  $\omega$ . Point  $P$  lies on  $\overline{EF}$  with  $\overline{DP} \perp \overline{EF}$ . Prove that lines  $SH$ ,  $MQ$ , and  $PL$  are concurrent.



Let  $\overline{LP}$  intersect  $\omega$  again at  $X$ . Since  $\overline{DHLA}$  bisects  $\angle FDE$ ,  $L$  is the midpoint of  $\widehat{EF}$  on  $\omega$ . Note that inversion at  $L$  with power  $LE^2 = LF^2$  swaps  $\overline{EF}$  and  $\omega$ , so it swaps  $(Q, D)$  and  $(P, X)$ . Thus,  $LQ \cdot LD = LE^2 = LP \cdot LX$ , so  $PQDX$  is cyclic and  $\angle QXD = \angle QPD = 90^\circ = \angle MXD$ , and  $X \in \overline{MQ}$ .

Let  $S'$  be the intersection of ray  $XH$  and  $\Gamma$ . It is sufficient to show that  $DA = DS'$ .

**Claim.**  $AXDS'$  is cyclic.

*Proof.* We present two proofs:

- Notice that the negative inversion at  $H$  with power  $-AH \cdot HD$  swaps  $(A, D)$ ,  $(B, E)$ , and  $(C, F)$ , so it swaps  $\Gamma$  and  $\omega$ . It is immediate that it swaps  $S'$  and  $X$ , so  $HX \cdot HS' = HA \cdot HD$ , as desired.
- Notice that homothety at  $H$  sends  $D$  and  $X$  to points  $D'$  and  $X'$  on  $\Gamma$ . Check that

$$HA \cdot HD = \frac{1}{2}HA \cdot HD' = \frac{1}{2}HS' \cdot HX' = HS' \cdot HX,$$

as desired. □

Since  $\angle AEH = 90^\circ$  and  $\overline{EH}$  bisects  $\angle DEQ$ ,  $-1 = (DQ; AH)$ . However,  $\angle DXQ = 90^\circ$ , so  $\overline{XD}$  bisects  $\angle AXH$  by a well-known lemma on Apollonius circles. It follows that  $D$  is the midpoint of one of the arcs  $\widehat{AS'}$  on  $(AXDS')$ , so  $DA = DS'$ , and  $S = S'$ , as desired.

## §2.4 Solutions to MOP Test 4

### Problem R4.1 (Putnam 2000 B1)

Let  $a_j, b_j, c_j$  be integers for  $1 \leq j \leq N$ . Assume for each  $j$ , at least one of  $a_j, b_j, c_j$  is odd. Show that there exist integers  $r, s, t$  such that  $ra_j + sb_j + tc_j$  is odd for at least  $\frac{4N}{7}$  values of  $j$ ,  $1 \leq j \leq N$ .

We use the probabilistic method. Take everything modulo 2.

**Claim.** For all  $(a_j, b_j, c_j) \in \mathbb{F}_2^3$  there are exactly 4 triples  $(r, s, t) \in \mathbb{F}_2^3$  such that  $ra_j + sb_j + tc_j$  is odd.

*Proof.* Omitted. □

By linearity of expectation, the expected number of  $j$  such that  $ra_j + sb_j + tc_j$  given a fixed triple  $(r, s, t)$  is  $\frac{4N}{7}$ , so there is at least one triple  $(r, s, t)$  that yields at least  $\frac{4N}{7}$  odd.

### Problem R4.2 (Tuymaada 2001/6)

Let  $n \geq 3$  be a fixed integer. An  $n \times n$  table of nonzero real numbers is given with the property that for any cell, the sum of the  $2n - 2$  numbers in the other cells of the same row or column is exactly  $k$  times the number in that cell. Determine all possible values of  $k$ , in terms of  $n$ .

The answer is  $-2$ ,  $n - 2$ , and  $2n - 2$ . It is easy to check that these constructions work:

- For  $k = -2$ , set a main diagonal all equal to  $-(n - 1)$ , and set everything else equal to 1.
- For  $k = n - 2$ , take the entire left column as  $-(n - 1)$ , and set everything else equal to 1.
- For  $k = 2n - 2$ , set all cells equal to 1.

We will now show these are the only possible values of  $k$ . Let  $\ell = k + 2$ , and assume that  $k \notin \{-2, n - 2\}$ ; that is,  $\ell \notin \{0, n\}$ . Denote by  $T$  the sum of all  $n^2$  elements, and let  $a_{i,j}$  denote the element in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column. This way, the given condition is equivalent to

$$\ell a_{p,q} = \sum_{i=1}^n a_{i,q} + \sum_{j=1}^n a_{p,j}. \quad (*)$$

Summing over all  $p$ ,

$$\ell \sum_{i=1}^n a_{i,q} = n \sum_{i=1}^n a_{i,q} + T \implies \sum_{i=1}^n a_{i,q} = \frac{T}{\ell - n}.$$

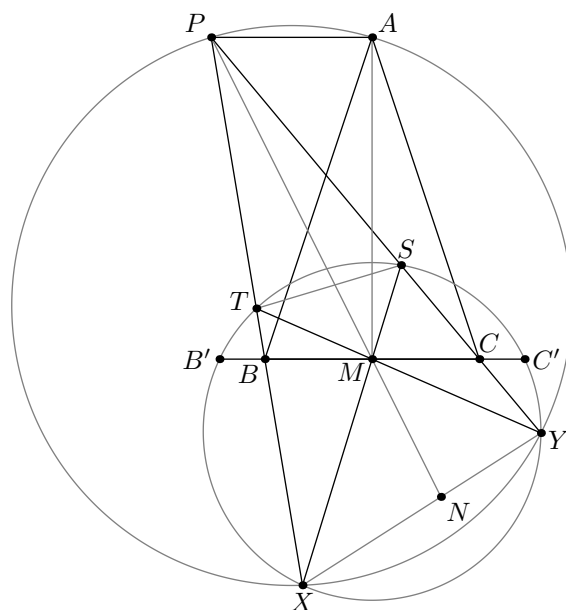
Hence, every row and column has a fixed sum  $S$ , so by  $(*)$ ,  $a_{p,q} = \frac{2S}{\ell}$ . Each cell is the same, so

$$\frac{2T}{\ell} = S = \frac{T}{\ell - n} \implies \ell = 2n \implies k = 2n - 2,$$

as requested.

### Problem R4.3 (ISL 2018 G2)

Let  $ABC$  be a triangle with  $AB = AC$ , and let  $M$  be the midpoint of  $\overline{BC}$ . Let  $P$  be a point such that  $PB < PC$  and  $\overline{PA} \parallel \overline{BC}$ . Let  $X$  and  $Y$  be points on lines  $PB$  and  $PC$  such that  $B$  lies inside segment  $PX$ ,  $C$  lies inside segment  $PY$ , and  $\angle PXM = \angle PYM$ . Prove that quadrilateral  $APXY$  is cyclic.



Let  $S = \overline{PC} \cap \overline{XM}$ ,  $T = \overline{PB} \cap \overline{YM}$ , and  $N = \overline{PM} \cap \overline{XY}$ . Note that  $\angle TXS = \angle PXM = \angle PYM = \angle SYT$ , so  $XYST$  is cyclic. Denote by  $O$  the circumcenter of  $XYST$ .

Let line  $BC$  intersect  $(XYST)$  at  $B'$  and  $C'$ . By the converse of the Butterfly Theorem,  $M$  is the midpoint of  $B'C'$ , so  $OM \perp BC$ . Thus  $O, M, A$  are collinear and  $OM \perp PA$ , whence  $A$  is the Miquel point of  $XYST$ , and the result follows.

**Problem R4.4 (ISL 2018 A5)**

Determine all functions  $f : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  satisfying

$$\left(x + \frac{1}{x}\right) f(y) = f(xy) + f\left(\frac{y}{x}\right)$$

for all  $x, y > 0$ .

The answer is  $f(x) = ax + \frac{b}{x}$  for real numbers  $a$  and  $b$ . It is easy to check that this works; so it suffices to prove that these are the only solutions.

Let  $P(x, y)$  denote the assertion. Notice that if  $f$  and  $g$  are solutions to the functional equation,  $f + g$  and  $f - g$  are also solutions. Given  $f(1)$  and  $f(2)$ , we can easily find  $a$  and  $b$  such that  $f(1) = a + b$  and  $f(2) = 2a + \frac{b}{2}$ . Shift down to give  $f(1) = f(2) = 0$ . We will show that  $f \equiv 0$ .

$P(2, 2)$  gives

$$\frac{5}{2}f(2) = f(4) + f(1) \implies f(4) = 0.$$

Now,  $P(2x, 2)$  and  $P(x, 1)$  give

$$f(4x) = -f\left(\frac{1}{x}\right) = f(x)$$

for all  $x$ , so  $P(2, 2x)$  gives

$$\frac{5}{2}f(2x) = f(4x) + f(x) = 2f(x) \implies f(2x) = \frac{4}{5}f(x) \implies f(x) = f(4x) = \frac{16}{25}f(x),$$

which immediately yields  $f \equiv 0$ , as desired.